TRANSFORMATION OF STRESS - MOHR'S CIRCLE
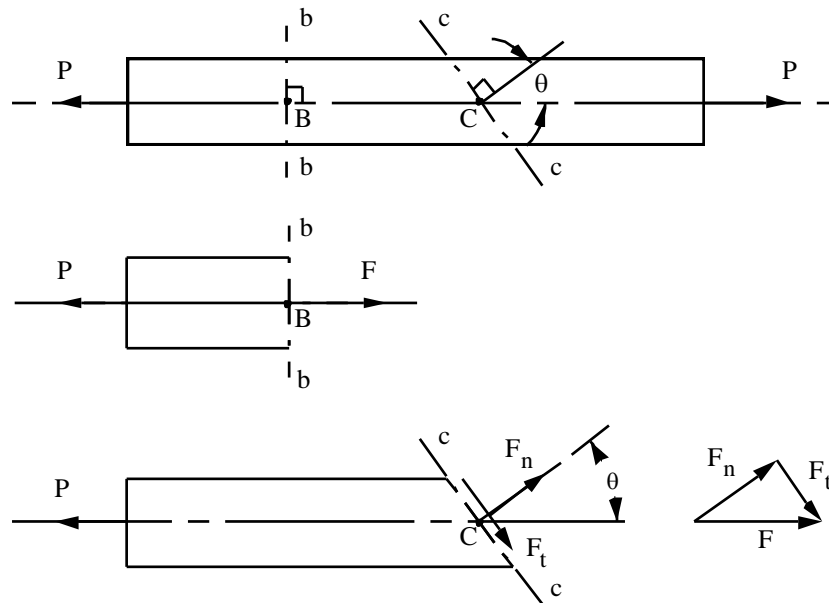
INTRODUCTION

Figure 1: Rod Subjected to Simple Tension

Figure 1 shows a rod subjected to simple tension by an axial force P. On section bb, the stress resultant is F and the average stress over bb is a direct stress given by:

$$\sigma\big|_{bb} = \frac{F}{\text{Area bb}} \quad (F = P)$$

On section cc, the stress resultants are F_n and F_t. F_n is the resultant of the direct stresses acting on plane cc and F_t is the resultant of the shear stresses acting on plane cc. Plane cc is thus subjected to both direct and shear stress where the average stresses are given by:

$$\sigma\big|_{cc} = \frac{F_n}{\text{Area cc}} \qquad \tau\big|_{cc} = \frac{F_t}{\text{Area cc}}$$

(Note that the vector sum of F_n and F_t is F.)

Point B, however, could also be point C. We must therefore conclude that at any point, the nature of the stress on a plane is a function of the orientation of that plane. This means that at any given point in a stressed body, different planes have different values of direct and shear stress. It is this difference which leads to ductile materials failing on 45° planes in simple tension (cup and cone failure) and brittle materials failing on 45° planes in simple torsion (helical failure).

When assessing the significance of the state of stress in a body it is thus important to consider the total state of stress at a point, not just the stress on one plane or element orientation.
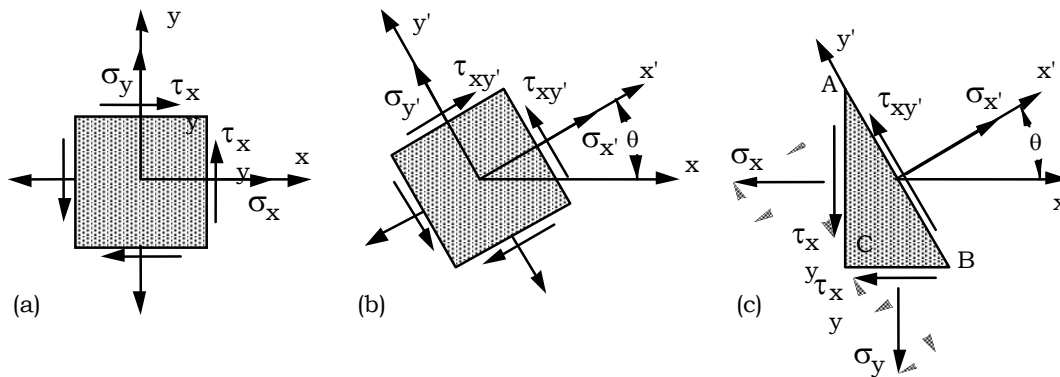
TRANSFORMATION OF STRESS

Figure 2: The General 2D State of Stress

Figure 2 shows the general two dimensional state of stress at a point. In Figure 2a, the stress components are shown on the xy planes and in Figure 2b, the stress components are shown on the xy' planes. These two figures represent the same total state of stress but the stress components are different on different planes.

Thus given the differential sensitivity of most materials to shear and direct stress, there is a need to be able to determine the stress components on any plane from the stress components on any other planes. i.e. given the stress on the xy planes, determine the stress on the xy' planes.

Figure 2c shows a triangular element with sides normal to the x, y and x' directions. Assuming that the element has unit width and that the elemental forces are given by (stress) x (area), equilibrium of the forces on the element leads to:

$$\sum F_{x'} = 0 = \sigma_{x'} AB - \sigma_x AC \cos\theta - \sigma_y BC \sin\theta - \tau_{xy} AC \sin\theta - \tau_{xy} BC \cos\theta$$

$$\sum F_{y'} = 0 = \tau_{xy'} AB + \sigma_x AC \sin\theta - \sigma_y BC \cos\theta - \tau_{xy} AC \cos\theta + \tau_{xy} BC \sin\theta$$

Putting AC = AB cosθ, BC = AB sinθ and then dividing by AB leads to:

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \,..................................(1)$$

$$\tau_{xy'} = -\sigma_x \sin\theta\cos\theta + \sigma_y \cos\theta\sin\theta + \tau_{xy} \cos^2\theta - \tau_{xy} \sin^2\theta \,.......(2)$$

Eqns. (1) and (2) can be simplified by the use of the identities:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta); \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta); \quad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta; \text{ giving:}$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \,..................................... (3)$$

$$\tau_{xy'} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \,.. (4)$$

MAXIMUM DIRECT STRESS

The orientation of the maximum (or minimum) direct stress is given by:

$$\frac{d\sigma_{x'}}{d\theta} = 0 = -\left(\sigma_x - \sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos\theta$$

which leads to:

$$\tan 2\theta = \frac{\pm 2\tau_{xy}}{\pm\left(\sigma_x - \sigma_y\right)} \dots\dots\dots (5)$$
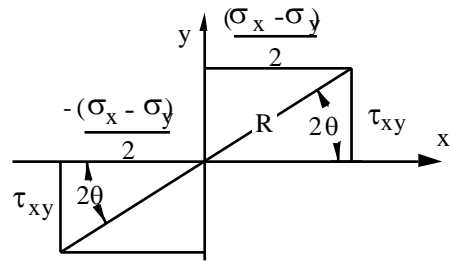
Equation (5) has two solutions; the angles which correspond to the maximum and minimum direct stress. Note also that the difference between these angles is 90° ($2\theta = 180°$). The maximum and minimum direct stresses thus act on mutually perpendicular planes.

Figure 3: Orientation of σ_{max}

From Eqn. (5) and Figure 3 we have:

$$\sin 2\theta = \pm\frac{\tau_{xy}}{R}; \quad \cos 2\theta = \pm\frac{\dfrac{\sigma_x - \sigma_y}{2}}{R} \quad \text{and} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Substituting for sin2θ and cos2θ in Eqns. (3) and (4) gives the maximum and minimum direct stress and the corresponding shear stress:

$$\left(\sigma_{x'}\right)_{max/\ min} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sigma_x - \sigma_y}{2}\frac{\dfrac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \pm \tau_{xy}\frac{\tau_{xy}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \frac{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \pm \frac{\tau_{xy}^2}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \dots\dots\dots\dots\dots\dots\dots (6)$$

$$\left(\tau_{xy'}\right)_{\sigma_{max/\ min}} = -\frac{\sigma_x - \sigma_y}{2}\frac{\pm\tau_{xy}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} + \tau_{xy}\frac{\pm\dfrac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$= 0$$

PRINCIPAL STRESSES

The maximum and minimum direct stresses are known as the *principal stresses* and principal stresses act on mutually perpendicular planes where the shear stress is zero. The planes on which the principal stresses act are known as the *principal planes* and the normals to these planes, the directions of the principal stresses, are the *principal axes*.
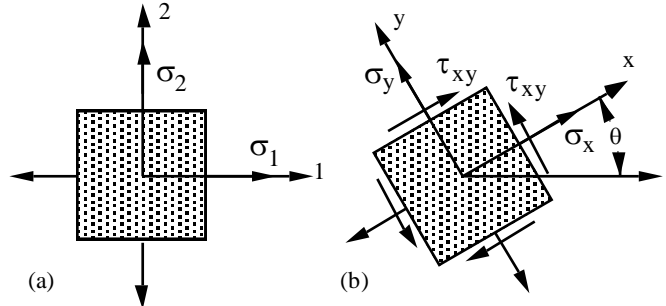

Figure 4: The General State of Stress From Principal Stresses

Figure 4 replicates Figure 2 with the principal stresses as the "reference" stress state and from of Eqns. (3) and (4), the stresses on the x, y planes are given by:

$$\sigma_{x,y} = \frac{\sigma_1 + \sigma_2}{2} \pm \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \quad \text{...} (7)$$

$$\tau_{xy} = -\frac{(\sigma_1 - \sigma_2)}{2} \sin 2\theta \quad \text{...} (8)$$

Note that in Figure 4, θ is the angle from the 1 axis to the x axis. The stress on the y plane is given by noting that the y axis is $\theta+90$ from the 1 axis and that $\cos2(\theta+90) = -\cos2\theta$.
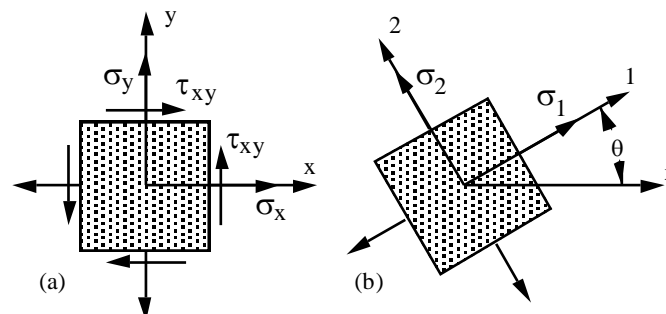

Figure 5: Principal Stresses From the General State of Stress

Eqns. (5) and (6) give the principal axes and stresses and axes from the general state of stress as set out below in Eqns. (9) and (10). Note that in these expressions, θ is the angle from the x axis to the 1 axis.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{.....................................} (9)$$

$$\tan 2\theta = \frac{\pm 2\tau_{xy}}{\pm(\sigma_x - \sigma_y)} \quad \text{...}(10)$$

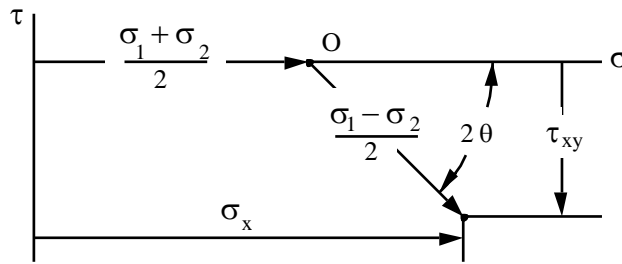## MOHR'S CIRCLE AND MAXIMUM SHEAR STRESS

Figure 6: Parametric Representation of Eqns. (7) and (8)

Figure 6 shows that Eqns. (7) and (8) represent the parametric equations of a circle known as *Mohr's Circle of Stress*. All direct and shear stress pairs for the state of stress at any point lie on Mohr's Circle.
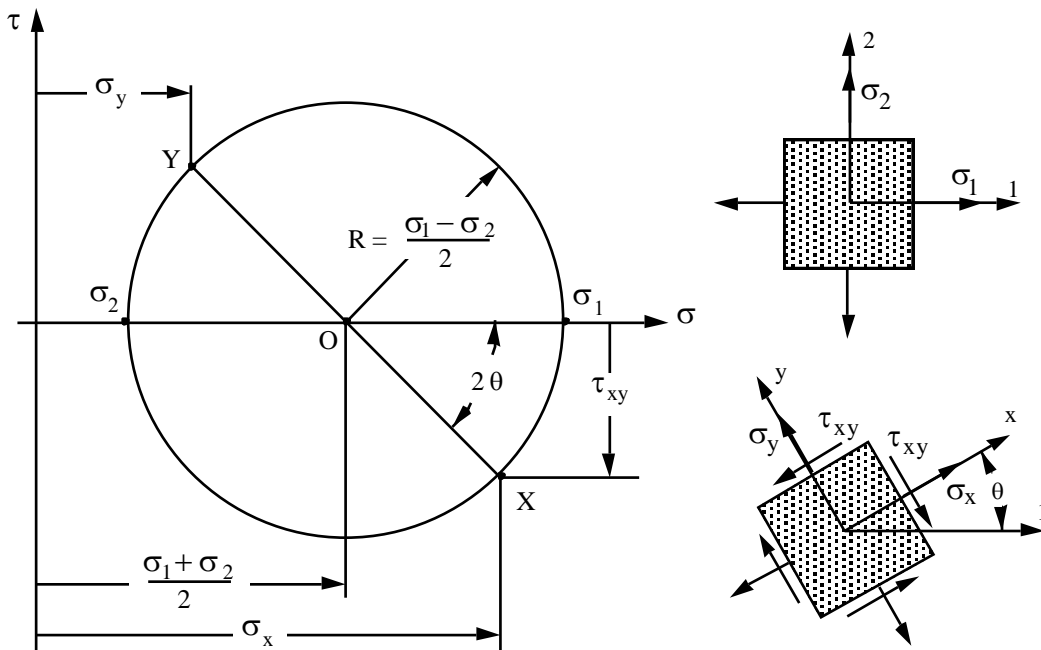

Figure 7: Mohr's Circle of Stress

From Figure 7 it can be seen that the maximum shear stress acts on planes which are ±45° from the principal planes and that the maximum shear stress is given by:

$$\tau_{max} = \frac{\left(\sigma_x - \sigma_y\right)}{2} \quad \text{......(12)}$$

On the planes of maximum shear the direct stresses are both equal to $(\sigma_x + \sigma_y)/2$. Mohr's Circle also shows that for all orthogonal planes, the sum of the direct stresses is a constant:
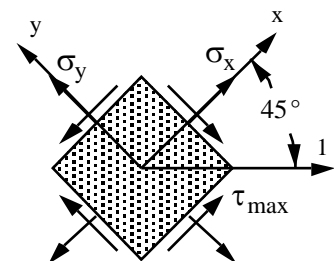
$$\sigma_x + \sigma_y = \sigma_1 + \sigma_2 \quad \text{......(13)}$$

Figure 8: Maximum Shear Stress

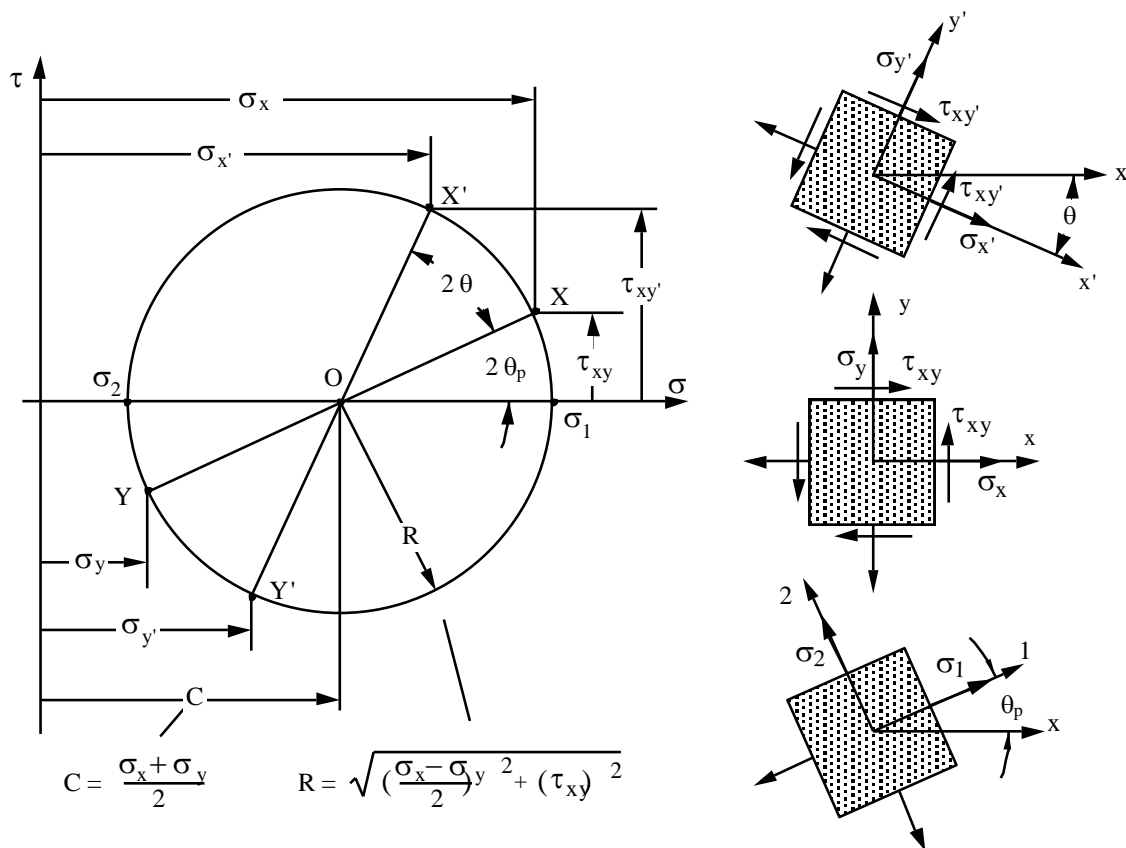## TRANSFORMATION OF STRESS USING MOHR'S CIRCLE

Figure 9: Mohr's Circle and Transformation of Stress

SUMMARY

1   τ_{max} occurs at $\pm 45°$ to the principal axes.

2      $\tau_{max} = (\sigma_1 - \sigma_2)/2$

3      $\sigma_x + \sigma_y = 2C = $ a constant.

4      $\sigma_x = \sigma_y = C$ when $\tau = \tau_{max}$.

5   A clockwise rotation from a reference axis on the body is an anticlockwise rotation from the corresponding reference point on Mohr's circle, and vice versa.

     e.g.

     i)   The x' axis is θ CW from x on the body but point X' is 2θ CCW from point X on Mohr's circle.

     ii)   The 1 axis is θ_p CCW from x on the body but σ_1 is $2\theta_p$ CW from point X on Mohr's circle.

Note: It is conventional to define $\sigma_1 > \sigma_2$.
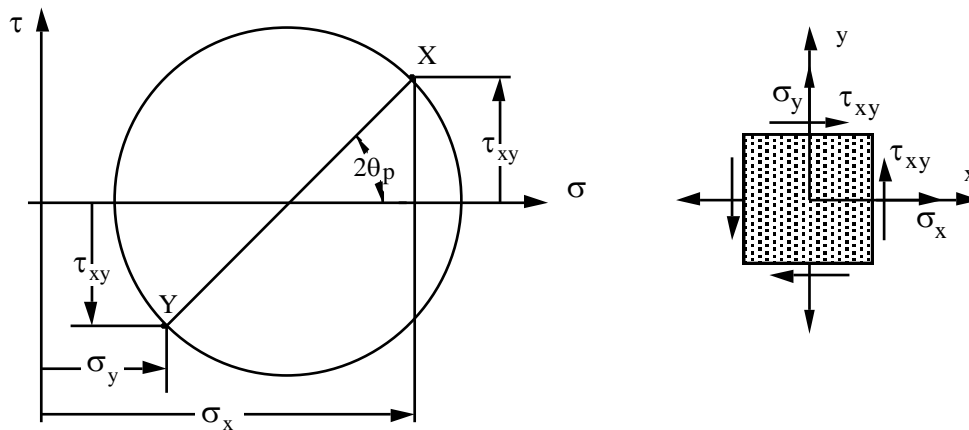
PROBLEM SOLVING

Figure 10: Mohr's Circle and the Sign of τ_{xy}

Figure 10, shows an element with its sides aligned with the xy planes and the shear stress on this element, τ_{xy}, is +ve. This agrees with the sign of the shear stress using point X on Mohr's circle but if point Y is used, the sign of τ_{xy} appears to be -ve. There would thus appear to be an ambiguity in the sign of τ. This is not so.

Figure 11a shows the principal stresses at the point represented by Figure 10 and Figure 11b shows the stresses on a plane at an angle of θ_p to the 1 principal axis. These stresses are represented by point X on Mohr's circle and on this plane, the shear stress is +ve. Point Y, however, describes the stress on a plane, whose normal is x' at an angle of $90° + \theta_p$ to the 1 principal axis. As shown in Fig11c, the shear stress on this plane, $\tau_{xy'}$, is -ve if the plane is described by axes, xy'.
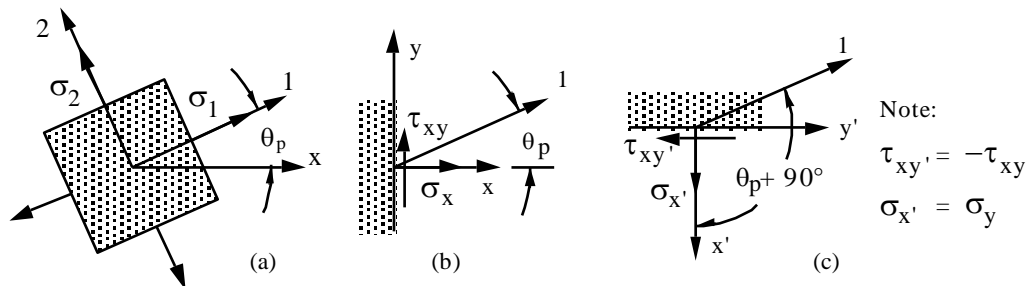

Figure 11: The Stresses on the Point X Plane and the Point Y Plane

There is thus no ambiguity in Mohr's circle as any point on the circle only represents one plane, not two orthogonal planes. A pair of diametral points will, however, give the stress on orthogonal planes. It is therefore suggested that analyses involving Mohr's circle be based on:

•   point X to determine the sign of the shear stress and σ_x

•   point Y to determine σ_y

•   point X and σ_1 to determine the relative orientation of the xy and 12 axes.

Figure 12 shows how the same state of stress is represented by different and arbitrary "reference" axes.
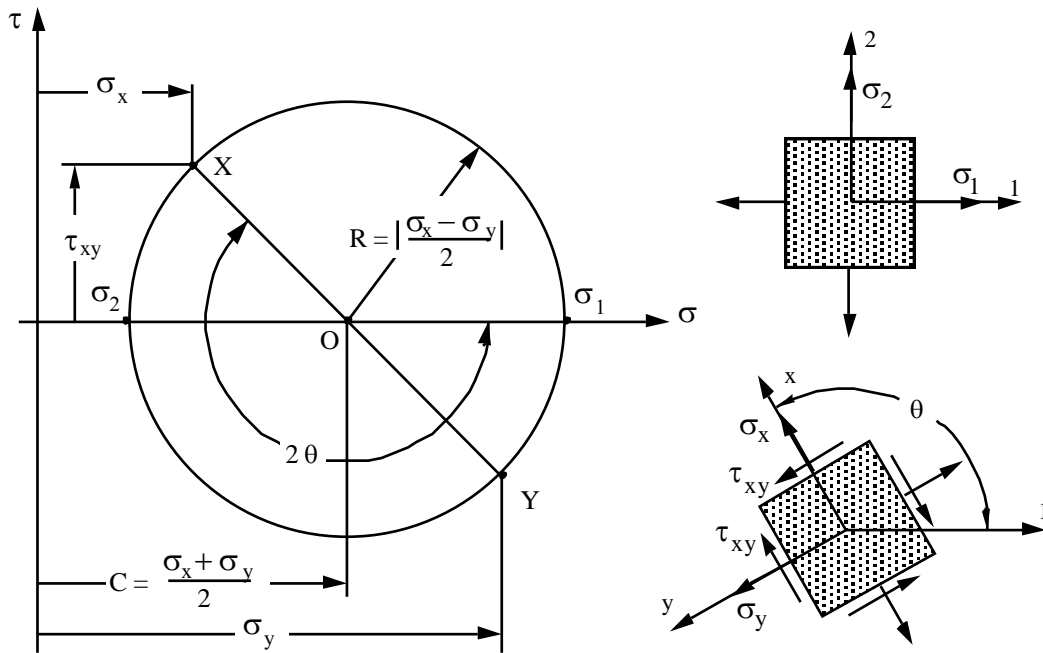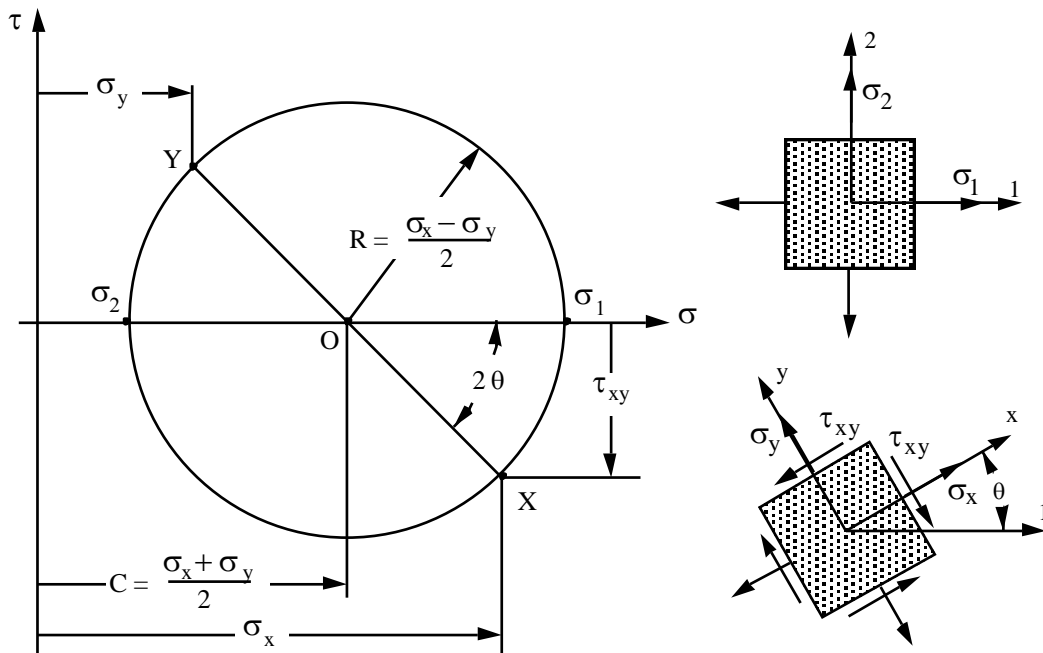
e.g.

Figure 12: A State of Stress Described by Two Different "x" Directions.